

Limits and Continuity

Big Ideas

Need limits to investigate instantaneous rate of change

Do not care what the function is actually doing at the point in question

Limits may exist at a point even if the function itself does not exist at that point

No reason to think that the limit will have the same value as the function at that point

Not all limits will exist

Limit is the height that the function intends to reach

Limits at a point
Limits at infinity
Infinite limits
Infinity is not a number, but rather indicates a direction

Vertical and horizontal asymptotes

Vertical asymptotes: \( \lim_{x \to a} f(x) = \pm\infty \) then \( x = a \) is a vertical asymptote

Horizontal asymptotes: \( \lim_{x \to \pm\infty} f(x) = a \) then \( y = a \) is a horizontal asymptote

One-sided limits

\( \lim_{x \to a^+} f(x) = L \) iff \( \lim_{x \to a^+} f(x) = L \) and \( \lim_{x \to a^-} f(x) = L \)

Continuity

\( \lim_{x \to a} f(x) = f(c) \) Must show three conditions are met

Discontinuity

Removable (limit exists)
Non-removable (limit does not exist: jump, infinite, oscillating)

Sandwich Theorem (Squeeze Theorem)

Indeterminate forms: \( \frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, -\infty, 1^\circ, 0^0, \infty^0 \)

Determinate Forms: \( \frac{k}{\infty} \to 0 \quad \frac{k}{0} \to \infty \quad \frac{\infty}{k} \to \infty \quad \frac{0}{k} \to 0 \quad \infty + \infty \to \infty \quad \infty\cdot\infty \to \infty \)

Properties of limits

End Behavior
AP Multiple Choice Questions

2008 AB Multiple Choice

Problems 1 5 77

2008 BC Multiple Choice

Problems 3 78

2003 AB Multiple Choice

3. For \( x \geq 0 \), the horizontal line \( y = 2 \) is an asymptote for the graph of the function \( f \). Which of the following statements must be true?

A) \( f(0) = 2 \) 
B) \( f(x) \neq 2 \) for all \( x \geq 0 \) 
C) \( f(2) \) is undefined 
D) \( \lim_{x \to 2} f(x) = \infty \) 
E) \( \lim_{x \to \infty} f(x) = 2 \)

6. \( \lim_{x \to \infty} \frac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} = \)

A) 4 
B) 1 
C) \( \frac{1}{4} \) 
D) 0 
E) -1

79. For which of the following does \( \lim_{x \to 4} f(x) \) exist?

A) I only 
B) II only 
C) III only 
D) I and II only 
E) I and III only
2003 BC Multiple Choice

81. The graph of the function $f$ is shown in the figure on the right. The value of $\lim_{x \to a} \sin(f(x))$ is

A) 0.909  B) 0.841  C) 0.141  D) -0.416  E) nonexistent

1998 AB Multiple Choice

26. The function $f$ is continuous on the closed interval $[0, 2]$ and has values that are given in the table on the right. The equation $f(x) = \frac{1}{2}$ must have at least two solutions in the interval $[0, 2]$ if $k =$

A) 0  B) $\frac{1}{2}$  C) 1  D) 2  E) 3

76. The graph of a function $f$ is shown in the figure on the right. Which of the following statements about $f$ is false?

A) $f$ is continuous at $x = a$
B) $f$ has a relative maximum at $x = a$
C) $x = a$ is in the domain of $f$
D) $\lim_{x \to a} f(x)$ is equal to $\lim_{x \to a} f(x)$
E) $\lim_{x \to a} f(x)$ exists
83. If \( a \neq 0 \), then \( \lim_{x \to a} \frac{x^4 - a^4}{x^2 - a^2} \) is

A) \( \frac{1}{a^2} \)  
B) \( \frac{1}{2a^2} \)  
C) \( \frac{1}{6a^2} \)  
D) 0  
E) nonexistent

1997 AB Multiple Choice

15. The graph of the function \( f \) is shown in the figure on the right. Which of the following statements about \( f \) is true?

A) \( \lim_{x \to a} f(x) = \lim_{x \to b} f(x) \)  
B) \( \lim_{x \to a} f(x) = 2 \)  
C) \( \lim_{x \to b} f(x) = 2 \)  
D) \( \lim_{x \to b} f(x) = 1 \)  
E) \( \lim_{x \to b} f(x) \) does not exist

21. \( \lim_{x \to 1} \frac{x}{\ln x} \) is

A) 0  
B) \( \frac{1}{e} \)  
C) 1  
D) \( e \)  
E) nonexistent
AP Free Response Questions

2012 AB 4 Part c (continuity)
The function \( f \) is defined by \( f(x) = \sqrt{25 - x^2} \) for \(-5 \leq x \leq 5\).

c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5 \end{cases} \).

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

2011 AB 6 Part a (continuity)
Let \( f \) be a function defined by 
\[
\begin{align*}
  f(x) &= \begin{cases} 
  1 - 2\sin x & \text{for } x \leq 0 \\
  e^{-x} & \text{for } x > 0 
  \end{cases}.
\end{align*}
\]

a) Show that \( f \) is continuous at \( x = 0 \).

1998 AB 2 Part a
Let \( f \) be the function given by \( f(x) = 2xe^{2x} \).

a) Find \( \lim_{x \to -\infty} f(x) \) and \( \lim_{x \to \infty} f(x) \).

1978 AB 3 Parts a b
Given the function \( f \) defined by 
\( f(x) = \frac{2x - 2}{x^2 + x - 2} \)

a) For what values of \( x \) is \( f(x) \) discontinuous?
b) At each point of discontinuity found in part (a), determine whether \( f(x) \) has a limit, and if so give the value of the limit.

1971 AB 6 Part c
A particle starts at the point \((5, 0)\) at \( t = 0 \) and moves along the x-axis in such a way that at time \( t > 0 \) its velocity \( v(t) \) is given by \( v(t) = \frac{t}{1 + t^2} \).

c) Find the limiting value of the velocity as \( t \) increased without bound.
Table of Questions

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Handouts

Limits and Continuity
AP Calculus

Chapter 2 – Section 1 Limits

1. The graphs of \( f \) and \( g \) are given below.

![Graph of f](image1)

![Graph of g](image2)

Determine whether the following limits exist. If they do, then find the limit.

a. \( \lim_{x \to -1} f(x) \)

b. \( \lim_{x \to 1} f(x) \)

c. \( \lim_{x \to -1} g(x) \)

d. \( \lim_{x \to 1} g(x) \)

e. \( \lim_{x \to -1} f(x) + g(x) \)

f. \( \lim_{x \to 0} 2f(x) + 3g(x) \)

g. \( \lim_{x \to -1} f(x)g(x) \)

h. \( \lim_{x \to 0} f(x)g(x) \)

i. \( \lim_{x \to 0^+} \frac{f(x)}{g(x)} \)

j. \( \lim_{x \to 0^+} \frac{g(x)}{f(x)} \)

k. \( \lim_{x \to -2^+} g(f(x)) \)

l. \( \lim_{x \to 1^-} f(g(x)) \)

2. The graphs of functions \( f \) and \( g \) are those given in problem 1 above. Determine whether the following limits exist and find the limit when it exists.

a. \( \lim_{x \to -1} f(x) \)

b. \( \lim_{x \to 1} f(x) \)

c. \( \lim_{x \to -1} g(x) \)

d. \( \lim_{x \to 1} g(x) \)

e. \( \lim_{x \to 0^-} f(x + 2) \)

f. \( \lim_{x \to 1} f(x^2) \)
### Answers

#### Problem 1

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#### Problem 2

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<td>f</td>
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Refer to the graph above to answer each of the following questions. If a limit does not exist explain why.

1. \( \lim_{{x \to a^-}} f(x) = \)  
2. \( \lim_{{x \to a^+}} f(x) = \)  
3. \( \lim_{{x \to a^0}} f(x) = \)

4. \( \lim_{{x \to a^-}} f(x) = \)  
5. \( \lim_{{x \to a^+}} f(x) = \)  
6. \( \lim_{{x \to a^0}} f(x) = \)

7. \( \lim_{{x \to b^-}} f(x) = \)  
8. \( \lim_{{x \to b^+}} f(x) = \)  
9. \( \lim_{{x \to b^0}} f(x) = \)

10. \( \lim_{{x \to b^-}} f(x) = \)  
11. \( \lim_{{x \to b^+}} f(x) = \)  
12. \( \lim_{{x \to b^0}} f(x) = \)

13. \( \lim_{{x \to c^-}} f(x) = \)  
14. \( \lim_{{x \to c^+}} f(x) = \)  
15. \( \lim_{{x \to c^0}} f(x) = \)

16. \( \lim_{{x \to d^-}} f(x) = \)  
17. \( \lim_{{x \to d^+}} f(x) = \)  
18. \( \lim_{{x \to d^0}} f(x) = \)

19. \( f(b) = \)  
20. \( f(d) = \)  
21. \( f(e) = \)  

22. How does the answer to number 19 compare to the answer to number 8?
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<td>16</td>
<td>m</td>
<td>17</td>
<td>Does not exist</td>
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<tr>
<td>19</td>
<td>m</td>
<td>20</td>
<td>0</td>
<td>21</td>
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<td>22</td>
<td>While (f(b) = m) the limit as (x) approaches (b) does not exist because the left and right hand limits are not equal.</td>
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Chapter 2 Section 1 Trigonometric Limits

Find the following limits involving trigonometric functions.

1. \[ \lim_{x \to 0} \frac{1 + \sin x}{1 + \cos x} \]

2. \[ \lim_{x \to 0} \frac{x}{\sin 3x} \]

3. \[ \lim_{x \to 0} \frac{\tan 2x}{x} \]

4. \[ \lim_{x \to 0} \frac{\sin 2x}{4x} \]

5. \[ \lim_{x \to 0} \frac{\sin^2 x}{3x} \]

6. \[ \lim_{x \to 0} \frac{x \cdot \sin x}{1 - \cos x} \]

7. \[ \lim_{x \to 0} \frac{x^2}{1 - \cos^2 x} \]

8. \[ \lim_{x \to 0} \frac{\sin(2x^2)}{x} \]
Answers:
1. \( \frac{1}{2} \)
2. \( \frac{1}{3} \)
3. 2
4. \( \frac{1}{2} \)
5. 0
6. 2
7. 1
8. 0

How to work:

1. \( \lim_{x \to 0} \frac{1 + \sin x}{1 + \cos x} = \frac{1 + 0}{1 + 1} = \frac{1}{2} \)

2. \( \lim_{x \to 0} \frac{x}{\sin 3x} = \lim_{x \to 0} \frac{3x}{\sin 3x} = \frac{3 \cdot 1}{3} = \frac{1}{3} \)

3. \( \lim_{x \to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} \frac{2 \cdot \sin 2x}{2 \cdot x \cdot \cos(2x)} = \frac{2}{1 \cdot 1} = 2 \)

4. \( \lim_{x \to 0} \frac{\sin 2x}{4x} = \lim_{x \to 0} \frac{2 \cdot \sin 2x}{2x} = \frac{2}{1 \cdot 1} = \frac{1}{2} \)

5. \( \lim_{x \to 0} \frac{\sin^2 x}{3x} = \lim_{x \to 0} \frac{\sin x \cdot \sin x}{x} = \frac{1}{3} \cdot \frac{1}{1} = 0 \)

6. \( \lim_{x \to 0} \frac{x \cdot \sin x}{1 - \cos x} = \lim_{x \to 0} \frac{x \cdot \sin x}{1 - \cos x} = \lim_{x \to 0} \frac{x \cdot \sin x \cdot (1 + \cos x)}{\sin^2 x} = \lim_{x \to 0} \frac{x(1 + \cos x)}{\sin x} = \lim_{x \to 0} \frac{x}{\sin x} (1 + \cos x) = 1 \cdot 2 = 2 \)

7. \( \lim_{x \to 0} \frac{x^2}{1 - \cos^2 x} = \lim_{x \to 0} \frac{x^2}{\sin^2 x} = \lim_{x \to 0} \frac{x}{\sin x} \cdot \frac{x}{\sin x} = 1 \cdot 1 = 1 \)

8. \( \lim_{x \to 0} \frac{\sin(2x^2)}{x} = \lim_{x \to 0} \frac{2x \cdot \sin(2x^2)}{2x} = \lim_{x \to 0} \frac{\sin(2x^2)}{2x^2} \cdot \frac{2x}{1} = 1 \cdot 0 = 0 \)
Chapter 2 Sections 1 – 3 Limits

1. Use the table feature of the calculator to fill in the table and guess the value of the limit of the function \( f(x) \).

\[
\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
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<td>1.001</td>
<td>0.999</td>
</tr>
<tr>
<td>1.0005</td>
<td>0.9995</td>
<td>1.0001</td>
<td>0.99999</td>
</tr>
</tbody>
</table>

2. Evaluate the limit assuming that \( \lim_{x \to 4} f(x) = 3 \) and \( \lim_{x \to 4} g(x) = 2 \).

\[
\lim_{x \to 4} f(x)g(x) \quad \lim_{x \to 4} (2f(x) + 5g(x))
\]

\[
\lim_{x \to 4} g(x) \quad \lim_{x \to 4} \frac{f(x) + 1}{3g(x) - 9}
\]

\[
\lim_{x \to 4} f(x)^2 \quad \lim_{x \to 4} \frac{g(x)}{x - 4}
\]

3. Evaluate the limits algebraically or state that the limit does not exist.

\[
\lim_{x \to 2} \frac{x - 3x + 2}{x - 2} \quad \lim_{x \to 2} \frac{x - 2}{x^3 - 4x}
\]

\[
\lim_{x \to 0} \frac{\sqrt{2 + x} - 2}{x} \quad \lim_{x \to 0} \frac{1}{3 + x} \frac{1}{3}
\]

\[
\lim_{x \to 0} \frac{(3a + x)^2 - 9a^2}{x} \quad \lim_{x \to 8} \frac{x^2 - 64}{x - 9}
\]
4. Limits involving trigonometric functions. Find the following limits or state that they do not exist.

\[
\lim_{x \to 0} \frac{\sin x \cos x}{x} \quad \lim_{x \to 0} \frac{\sin(6x)}{x}
\]

\[
\lim_{x \to 0} \frac{\sin(7x)}{3x} \quad \lim_{x \to 0} \frac{\sin(2x)}{\sin(5x)}
\]

\[
\lim_{x \to 0} \frac{1}{\sin x} \quad \lim_{x \to 0} \frac{\sin(3x)\sin(2x)}{x\sin(5x)}
\]

5. Limits involving infinity. Find the following limits. Indicate if the function goes to \( \pm \infty \).

\[
\lim_{x \to 4} \frac{2x}{x - 4} \quad \lim_{x \to 2} 2\int(x) - 3
\]

\[
\lim_{x \to \infty} \frac{x^2 + 3}{x - 4} \quad \lim_{x \to \infty} \frac{x + \sin(x)}{2x}
\]

6. Use limits to find the vertical asymptote(s) of the function \( f(x) = \frac{x^2 + x - 6}{x^2 - x - 2} \).

7. Use limits to find the horizontal asymptote(s) of the function \( f(x) = \frac{3x - 5}{|2x + 4|} \).

8. Determine the points at which the function is discontinuous and state the type of discontinuity: removable, jump, infinite, or none of these.

\[
f(x) = \frac{1}{x - 2} \quad f(x) = \frac{x - 3}{x^2 - 9}
\]

\[
f(x) = \frac{x - 2}{|x - 1|} \quad f(x) = \frac{x - 2}{|x - 2|}
\]
9. Is the function \( f(x) \) continuous? Give reasons for your answer.
\[
f(x) = \begin{cases} 
  x + 1 & x < 1 \\
  \frac{1}{x} & x \geq 1
\end{cases}
\]

10. Find the value of \( c \) that makes the function continuous.
\[
f(x) = \begin{cases} 
  x^2 - c & x < 5 \\
  4x + 2c & x \geq 5
\end{cases}
\]

11. Intermediate Value Theorem

Use the IVT to show that \( f(x) = x^3 + x \) takes on the value 9 for some \( x \) in \([1, 2]\).

The temperature in Kansas City was 46\(^{\circ}\) at 6:00 AM and rose to 68\(^{\circ}\) at noon. What must we assume about temperature to conclude that the temperature was 60\(^{\circ}\) at some moment between 6:00AM and noon.

12. Draw the graphs of a function on \([0, 5]\) with the given properties.

\[
\begin{align*}
\lim_{x \to 1} f(x) &= 2, \quad \lim_{x \to 3} f(x) = 0, \quad \lim_{x \to 3^+} f(x) = 4 \\
\lim_{x \to -1} f(x) &= \infty, \quad \lim_{x \to 1} f(x) = 0, \quad \lim_{x \to 3} f(x) = -\infty \\
\lim_{x \to 2} f(x) &= f(2) = 3, \quad \lim_{x \to 2^-} f(x) = -1, \lim_{x \to 4^-} f(x) = 2 \neq f(4)
\end{align*}
\]

\( f(x) \) has a removable discontinuity at \( x = 1 \), a jump discontinuity at \( x = 2 \), and \( \lim_{x \to 3} f(x) = -\infty, \lim_{x \to 3^-} f(x) = 2 \)
1. A curve has equation $y = f(x)$.
Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(3+h, f(3+h))$.

Write an expression for the slope of the tangent line at $P$.

2. Consider the slope of the given curve at each of the five points shown.
List these five slopes in decreasing order and explain your reasoning.

3. Shown at the right are the position functions of two runners, $A$ and $B$, who run a 100-meter race and finish in a tie.
Runner $A$ is the top graph and runner $B$ is the lower graph.

Describe and compare how the runners run the race.

At what time is the distance between the runners the greatest?

At what time do they have the same speed?
In Problems 4 to 7, use the formula \( m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \) to find the slope of the tangent line.

4. Find the slope of the tangent to the parabola \( y = x^2 + 2x \) at the point \((-3, 3)\).
   Use the results to write the equation of the tangent line.

5. Find the slope of the tangent to the curve \( y = 3 + 4x - 2x^2 \) at the point \((2, 3)\).
   Find the equation of the tangent line.
   Find the equation of the normal line.

6. Find the slope of the tangent to the curve \( y = \frac{1}{2x} \) at the point \((2, \frac{1}{4})\).
   Find the equation of the tangent line.

7. Find the equation of the tangent to the curve \( f(x) = 2 - 3x^2 \) when the slope of the tangent is 12.

8. Find the average rate of change of the function \( f(x) = \sqrt{x} + 2 \) over the interval \([2, 23]\).

9. An object dropped from the top of a 200-foot cliff will fall \( s = 16t^2 \) feet in \( t \) seconds.
   How far will the object fall in 3 seconds?

   What is the average speed over these 3 seconds?

   What is the instantaneous speed at \( t = 3 \) seconds?